Magnetically Coupled Circuits

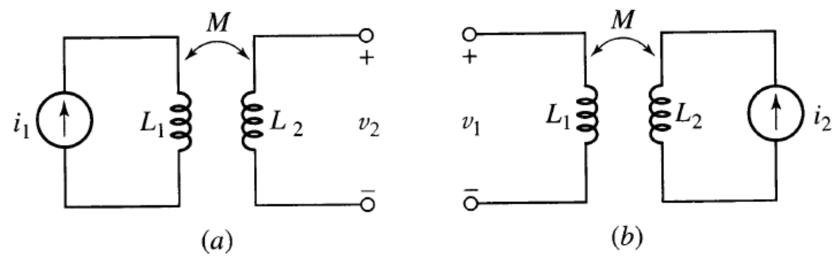
- Whenever a current flows through a conductor, a magnetic field is generated (magnetic flux)
- When time varying magnetic field generated by one loop penetrates a second loop, a voltage induced between the ends of the second wire
- New term is defined "mutual inductance" to differentiate from "inductance"

MUTUAL INDUCTANCE

Physical basis

- Production of magnetic flux by a current
- The production of voltage by time varying magnetic field
- Current flowing in one coil established flux about that coil and about second coil nearby
- The time varying flux surrounding second coil produces a voltage across the terminals of the second coil
- This voltage is proportional to time rate of change of current in first coil
- Define coefficient of mutual inductance or simply mutual inductance

$$v_2(t) = M_{21}\frac{di_1(t)}{dt}$$



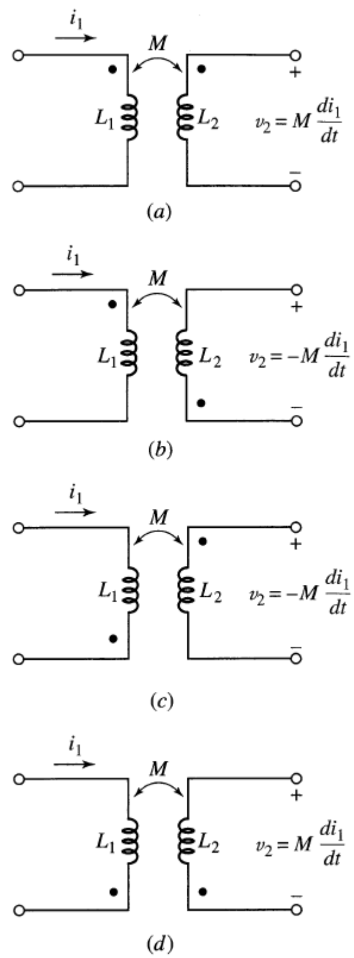
■ **FIGURE 13.1** (*a*) A current i_1 through L_1 produces an open-circuit voltage v_2 across L_2. (*b*) A current i_2 through L_2 produces an open-circuit voltage v_1 across L_1.

$$v_1(t) = M_{12}\frac{di_2(t)}{dt}$$

- $M_{12}=M_{21}=M$

Dot Convention

- A current entering the dotted terminal of one coil produces an open circuit voltage with positive voltage reference at the dotted terminal of the second coil
- A current entering the undotted terminal of one coil provides a voltage that is positively sensed at the undotted terminal of the second coil



Example

EXAMPLE **13.1**

For the circuit shown in Fig. 13.3, (*a*) determine v_1 if $i_2 = 5 \sin 45t$ A and $i_1 = 0$; (*b*) determine v_2 if $i_1 = -8e^{-t}$ A and $i_2 = 0$.

(*a*) Since the current i_2 is entering the *undotted* terminal of the right coil, the positive reference for the voltage induced across the left coil is the undotted terminal. Thus, we have an open-circuit voltage

$$v_1 = -(2)(45)(5 \cos 45t) = -450 \cos 45t \text{ V}$$

appearing across the terminals of the left coil as a result of the time-varying magnetic flux generated by i_2 flowing into the right coil.

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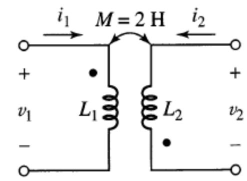


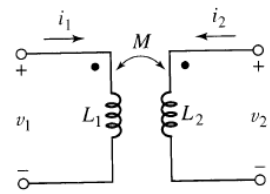
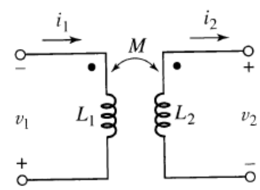
■ **FIGURE 13.3** The dot convention provides a relationship between the terminal at which a current enters one coil, and the positive voltage reference for the other coil.

Since no current flows through the coil on the left, there is no contribution to v_1 from self-induction.

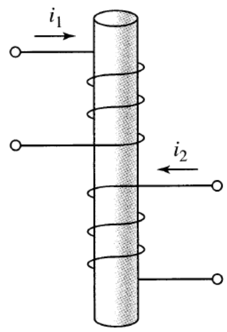
(*b*) We now have a current entering a *dotted* terminal, but v_2 has its positive reference at the *undotted* terminal. Thus,

$$v_2 = -(2)(-1)(-8e^{-t}) = -16e^{-t} \text{ V}$$

Combined Mutual and Self-Induction Voltage

$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$ $v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$	
$v_1 = -L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$ $v_2 = -L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$ For sinusoidal source $\mathbf{V}_1 = -j\omega L_1 \mathbf{I}_1 + j\omega M \mathbf{I}_2$ $\mathbf{V}_2 = -j\omega L_2 \mathbf{I}_2 + j\omega M \mathbf{I}_1$	

Physical Basis of the Dot Convention

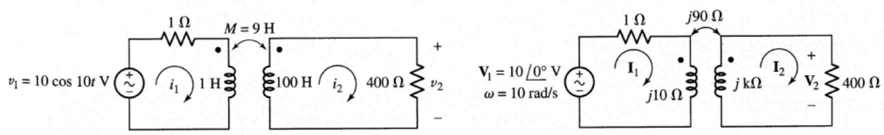


- The current i1 produces a flux which directed downward
- Since i1 is increasing with time, the flux, which is proportional to i1 is also increasing with time
- If i2 is a positive and increasing, i2 produces a magnetic flux which is directed downward and increasing
- The voltage across the terminals of any coil results from the time rate of change of the flux within the coil
- The voltage is greater with i2 flowing than it would be if i2 were zero

Example

EXAMPLE **13.2**

For the circuit shown in Fig. 13.7*a*, find the ratio of the output voltage across the 400 Ω resistor to the source voltage, expressed using phasor notation.



■ **FIGURE 13.7** (*a*) A circuit containing mutual inductance in which the voltage ratio $\mathbf{V}_2/\mathbf{V}_1$ is desired. (*b*) Self and mutual inductances are replaced by the corresponding impedances.

▶ ***Identify the goal of the problem.***
We need a numerical value for $\mathbf{V}_2$. We will then divide by $10\angle 0°$ V.

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▶ ***Collect the known information.***
We begin by replacing the 1 H and 100 H by their corresponding impedances, $j10\ \Omega$ and $j\ \text{k}\Omega$, respectively (Fig. 13.7*b*). We also replace the 9 H mutual inductance by $j\omega M = j90\ \Omega$.

▶ ***Devise a plan.***
Mesh analysis is likely to be a good approach, as we have a circuit with two clearly defined meshes. Once we find $\mathbf{I}_2$, $\mathbf{V}_2$ is simply $400\ \mathbf{I}_2$.

▶ ***Construct an appropriate set of equations.***
In the left mesh, the sign of the mutual term is determined by applying the dot convention. Since $\mathbf{I}_2$ enters the undotted terminal of L_2, the mutual voltage across L_1 must have the positive reference at the undotted terminal. Thus,

$$(1 + j10)\mathbf{I}_1 - j90\mathbf{I}_2 = 10\angle 0°$$

Since $\mathbf{I}_1$ enters the dot-marked terminal, the mutual term in the right mesh has its (+) reference at the dotted terminal of the 100 H inductor. Therefore, we may write

$$(400 + j1000)\mathbf{I}_2 - j90\mathbf{I}_1 = 0$$

▶ ***Determine if additional information is required.***
We have two equations in two unknowns, $\mathbf{I}_1$ and $\mathbf{I}_2$. Once we solve for the two currents, the output voltage $\mathbf{V}_2$ may be obtained by multiplying $\mathbf{I}_2$ by 400 Ω.

▶ ***Attempt a solution.***
Upon solving these two equations with a scientific calculator, we find that

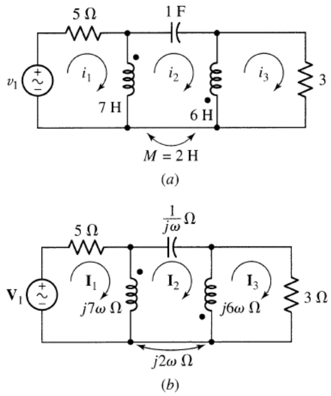
$$\mathbf{I}_2 = 0.172\angle -16.70°\ \text{A}$$

Thus,

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{400(0.172\angle -16.70°)}{10\angle 0°} = 6.880\angle -16.70°$$

Example

Write a complete set of phasor equations for the circuit of Fig. 13.10*a*.



■ **FIGURE 13.10** (*a*) A three-mesh circuit with mutual coupling. (*b*) The 1 F capacitance as well as the self- and mutual inductances are replaced by their corresponding impedances.

The circuit contains three meshes, and the three mesh currents have already been assigned. Once again, our first step is to replace both the mutual inductance and the two self-inductances with their corresponding impedances as shown in Fig. 13.10*b*. Applying Kirchhoff's voltage law to the first mesh, a positive sign for the mutual term is assured by selecting $(\mathbf{I}_3 - \mathbf{I}_2)$ as the current through the second coil. Thus,

$$5\mathbf{I}_1 + 7j\omega(\mathbf{I}_1 - \mathbf{I}_2) + 2j\omega(\mathbf{I}_3 - \mathbf{I}_2) = \mathbf{V}_1$$

or

$$(5 + 7j\omega)\mathbf{I}_1 - 9j\omega\mathbf{I}_2 + 2j\omega\mathbf{I}_3 = \mathbf{V}_1 \qquad [3]$$

The second mesh requires two self-inductance terms and two mutual-inductance terms; the equation cannot be written carelessly. We obtain

$$7j\omega(\mathbf{I}_2 - \mathbf{I}_1) + 2j\omega(\mathbf{I}_2 - \mathbf{I}_3) + \frac{1}{j\omega}\mathbf{I}_2 + 6j\omega(\mathbf{I}_2 - \mathbf{I}_3)$$
$$+ 2j\omega(\mathbf{I}_2 - \mathbf{I}_1) = 0$$

or

$$-9j\omega\mathbf{I}_1 + \left(17j\omega + \frac{1}{j\omega}\right)\mathbf{I}_2 - 8j\omega\mathbf{I}_3 = 0 \qquad [4]$$

Finally, for the third mesh,

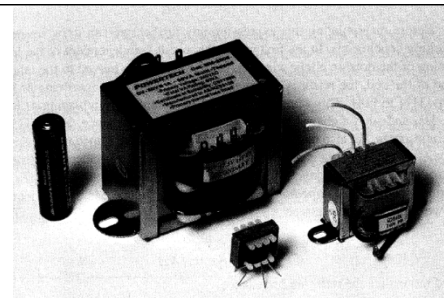
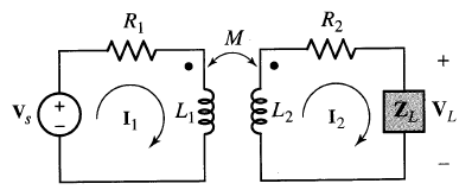
$$6j\omega(\mathbf{I}_3 - \mathbf{I}_2) + 2j\omega(\mathbf{I}_1 - \mathbf{I}_2) + 3\mathbf{I}_3 = 0$$

or

$$2j\omega\mathbf{I}_1 - 8j\omega\mathbf{I}_2 + (3 + 6j\omega)\mathbf{I}_2 = 0 \qquad [5]$$

Equations [3] to [5] may be solved by any of the conventional methods.

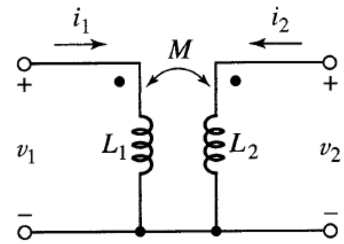
LINEAR TRANSFORMER

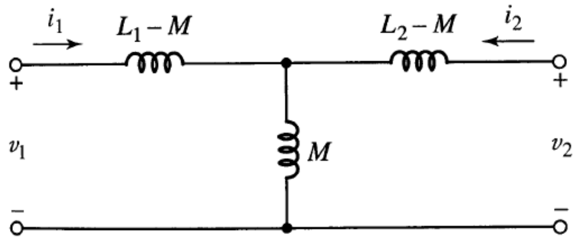
Primary Secondary	
Reflected Impedance $\mathbf{V}_s = (R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M \mathbf{I}_2$ $0 = -j\omega M \mathbf{I}_1 + (R_2 + j\omega L_2 + \mathbf{Z}_L)\mathbf{I}_2$ Define $\mathbf{Z}_{11} = R_1 + j\omega L_1$ and $\mathbf{Z}_{22} = R_2 + j\omega L_2 + \mathbf{Z}_L$ $\mathbf{V}_s = \mathbf{Z}_{11}\mathbf{I}_1 - j\omega M \mathbf{I}_2$ $0 = -j\omega M \mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2$ $\mathbf{Z}_{\text{in}} = \frac{\mathbf{V}_s}{\mathbf{I}_1} = \mathbf{Z}_{11} - \frac{(j\omega)^2 M^2}{\mathbf{Z}_{22}}$ $\mathbf{Z}_{\text{in}} = \mathbf{Z}_{11} + \frac{\omega^2 M^2}{R_{22} + jX_{22}}$ $\mathbf{Z}_{\text{in}} = \mathbf{Z}_{11} + \frac{\omega^2 M^2 R_{22}}{R_{22}^2 + X_{22}^2} + \frac{-j\omega^2 M^2 X_{22}}{R_{22}^2 + X_{22}^2}$ • Presence of secondary increases losses in primary	

T Equivalent Network

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$





T equivalent

Example

Find the T equivalent of the linear transformer shown in Fig. 13.19*a*.

We identify $L_1 = 30$ mH, $L_2 = 60$ mH, and $M = 40$ mH, and note that the dots are both at the upper terminals, as they are in the basic circuit of Fig. 13.17.

Hence, $L_1 - M = -10$ mH is in the upper left arm, $L_2 - M = 20$ mH is at the upper right, and the center stem contains $M = 40$ mH. The complete equivalent T is shown in Fig. 13.19*b*.

To demonstrate the equivalence, let us leave terminals *C* and *D* open-circuited and apply $v_{AB} = 10\cos 100t$ V to the input in Fig. 13.19*a*. Thus,

$$i_1 = \frac{1}{30 \times 10^{-3}} \int 10\cos(100t)\,dt = 3.33\sin 100t \text{ A}$$

and

$$v_{CD} = M\frac{di_1}{dt} = 40 \times 10^{-3} \times 3.33 \times 100\cos 100t$$
$$= 13.33\cos 100t \qquad \text{V}$$

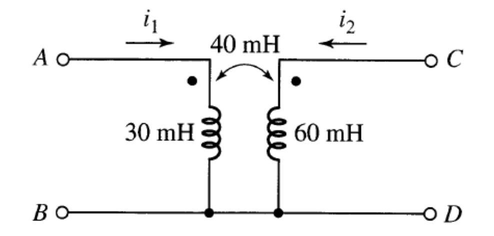
Applying the same voltage in the T equivalent, we find that

$$i_1 = \frac{1}{(-10 + 40) \times 10^{-3}} \int 10\cos(100t)\,dt = 3.33\sin 100t \qquad \text{A}$$

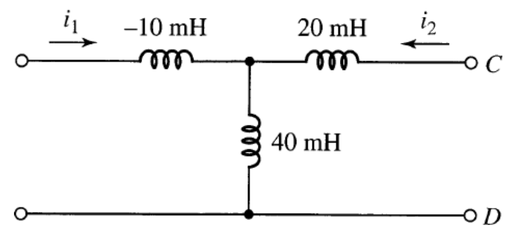
once again. Also, the voltage at *C* and *D* is equal to the voltage across the 40 mH inductor. Thus,

$$v_{CD} = 40 \times 10^{-3} \times 3.33 \times 100\cos 100t = 13.33\cos 100t \qquad \text{V}$$

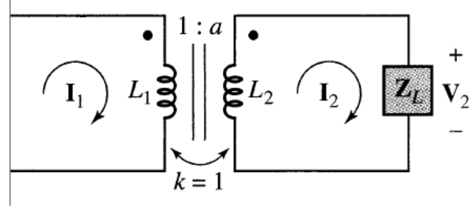
and the two networks yield equal results.



(*a*)



IDEAL TRANSFORMER

Ideal transformer is a useful approximation of a very tightly coupled transformer in which coupling coefficient is unity and both secondary and primary inductive reactance are extremely large	
Turns ratio $\frac{L_2}{L_1} = \frac{N_2^2}{N_1^2} = a^2$ $a = \frac{N_2}{N_1}$	
Sinusoidal Steady State $\mathbf{V}_1 = j\omega L_1 \mathbf{I}_1 - j\omega M \mathbf{I}_2$ $0 = -j\omega M \mathbf{I}_1 + (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2$ $\mathbf{V}_1 = \mathbf{I}_1 j\omega L_1 + \mathbf{I}_1 \frac{\omega^2 M^2}{\mathbf{Z}_L + j\omega L_2}$ $\mathbf{Z}_{in} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = j\omega L_1 + \frac{\omega^2 M^2}{\mathbf{Z}_L + j\omega L_2}$ $\mathbf{Z}_{in} = j\omega L_1 + \frac{\omega^2 L_1 L_2}{\mathbf{Z}_L + j\omega L_2}$	

$L_2 = a^2 L_1$ $\mathbf{Z}_{\text{in}} = j\omega L_1 + \frac{\omega^2 a^2 L_1^2}{\mathbf{Z}_L + j\omega a^2 L_1}$ $\mathbf{Z}_{\text{in}} = \frac{j\omega L_1 \mathbf{Z}_L - \omega^2 a^2 L_1^2 + \omega^2 a^2 L_1^2}{\mathbf{Z}_L + j\omega a^2 L_1}$ $\mathbf{Z}_{\text{in}} = \frac{j\omega L_1 \mathbf{Z}_L}{\mathbf{Z}_L + j\omega a^2 L_1} = \frac{\mathbf{Z}_L}{\mathbf{Z}_L / j\omega L_1 + a^2}$ As L_1 -> infinity $\mathbf{Z}_{\text{in}} = \frac{\mathbf{Z}_L}{a^2}$	

Use of Transformers for Impedance Matching

Example:

Match amplifier internal impedance of 4000Ω and a speaker impedance of 8Ω.

$$\mathbf{Z}_g = 4000 = \frac{\mathbf{Z}_L}{a^2} \doteq \frac{8}{a^2}$$

$$a = \frac{1}{22.4}$$

$$\frac{N_1}{N_2} = 22.4$$

Primary and secondary current relationship

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{j\omega M}{\mathbf{Z}_L + j\omega L_2}$$

If L_2 is very large and since $(M=(L_1L_2)^{1/2})$

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{j\omega M}{j\omega L_2} = \sqrt{\frac{L_1}{L_2}}$$

$$\boxed{\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{1}{a}}$$

$$N_1\mathbf{I}_1 = N_2\mathbf{I}_2$$

Use of Transformers for Voltage Level Adjustment

$$\mathbf{V}_2 = \mathbf{I}_2\mathbf{Z}_L$$

$$\mathbf{V}_1 = \mathbf{I}_1\mathbf{Z}_{\text{in}} = \mathbf{I}_1\frac{\mathbf{Z}_L}{a^2}$$

$$\mathbf{V}_1 = \mathbf{I}_1\mathbf{Z}_{\text{in}} = \mathbf{I}_1\frac{\mathbf{Z}_L}{a^2}$$

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = a^2\frac{\mathbf{I}_2}{\mathbf{I}_1}$$

$$\boxed{\frac{\mathbf{V}_2}{\mathbf{V}_1} = a = \frac{N_2}{N_1}}$$

$$\mathbf{V}_2\mathbf{I}_2 = \mathbf{V}_1\mathbf{I}_1$$

$$\mathbf{Z}_L = |\mathbf{Z}_L|\underline{/\theta}$$